2 Weighted Residual Methods

Fundamental equations Consider the problem governed by the differential equation:

\[ \Gamma u = -f \quad \text{in } D. \quad (83) \]

The above differential equation is solved by using the boundary conditions given as follows:

\[ u = \bar{u} \quad \text{on } S_u \]
\[ t = Bu = t \quad \text{on } S_t = S \setminus S_i. \quad (84) \]

where the boundary \( S \) consists of \( S_u \) and \( S_t \). The boundary conditions given in eqs. (84) and (85) are called rigid and natural boundary conditions, or Dirichlet and Neumann boundary conditions, respectively. In most engineering problems, \( \Gamma \) and \( B \) are differential operators in the forms of \( \Gamma = \nabla \cdot \Sigma \) and \( B = n \cdot \Sigma \), where \( \Sigma \) is another differential operator and \( n \) is the normal vector on \( S_t \).

For the Laplace problem, the governing equation is given by

\[ \nabla^2 u = -f \quad \text{in } D, \quad (86) \]

and the Neumann boundary condition on \( S_t \) is given as \( t = n \cdot \nabla u = \bar{t} \). Then the differential operators \( \Gamma, B \) and \( \Sigma \) are defined by \( \Gamma = \nabla^2 \), \( B = n \cdot \nabla \) and \( \Sigma = \nabla \), respectively.

Trial functions Suppose that \( u \) is approximated by trial function \( \tilde{u} \) of

\[ \tilde{u}(x) = \sum_{j=1}^{M} N^J(x)\tilde{u}^J \quad (87) \]

where \( \tilde{u}^J \) are coefficients and \( N^J(x) \) are linearly independent functions called trial bases, test functions or shape functions.

Weighted residual Since the trial function \( \tilde{u} \) does not satisfy the governing equation and the boundary condition rigorously, the residuals defined in the following are nonzero

\[ \begin{align*}
R_D(x) &= -\{\Gamma \tilde{u}(x) + f(x)\} \neq 0 \quad \text{in } D \quad (88) \\
R_S(x) &= \tilde{u} - \bar{u}(x) \neq 0 \quad \text{on } S_u, \quad \text{or } Bu(x) - \bar{t}(x) \neq 0 \quad \text{on } S_t. \quad (89)
\end{align*} \]

The problem is to find \( \tilde{u} \) to make zero the residuals weighted by the appropriate function \( w(x) \) as follows:

\[ \int_D w(x) \cdot R_D(x) dV + \int_S w(x) \cdot R_S(x) dS = 0 \quad (90) \]

where \( w(x) \) is called the weight function. Eq. (90) is also called a weak form to the original differential equation (83), called a strong form, because the solution that satisfies eq. (90) does not always satisfy the original differential equation (83) and the boundary condition (85).

Substitution of eq. (87) into eq. (90) yields the following equation:

\[ \begin{aligned}
\int_D w_I \cdot R_D dV + \int_S w_I \cdot R_S dS &= \int_D w_I(x) \cdot (-\{\Gamma N^J(x)\tilde{u}^J - f(x)\}) dV \\
&\quad + \int_{S_u} w_I(x) \cdot (N^J(x)\tilde{u}^J - \bar{u}(x)) dS + \int_{S_t} w_I(x) \cdot (B N^J(x)\tilde{u}^J - \bar{t}(x)) dS \\
&= \left\{ \int_D w_I(x) \cdot \Gamma N^J(x) dV + \int_{S_u} w_I(x)N^J(x) dS + \int_{S_t} w_I(x) \cdot BN^J(x) dS \right\} \tilde{u}^J \\
&\quad - \left\{ \int_D w_I(x) \cdot f(x) dV + \int_{S_u} w_I(x) \bar{u}(x) dS + \int_{S_t} w_I(x) \cdot \bar{t}(x) dS \right\} \int_{f_I}
\end{aligned} \]

\[ = 0 \quad (91) \]
where \( M \) independent weight functions \( w_I \) \((I = 1, 2, \ldots, M)\) are chosen. This equation can be solved for \( \hat{u}^J \). As shown in the sequel, there are several residual methods depending on the selection of weight functions.

It is not difficult to find trial functions \( \tilde{u} \) satisfying the boundary condition \( \tilde{u} = \bar{u} \) on \( S_u \). In such a case, no residual exists on \( S_u \) and the integrals on \( S_u \) disappear in eq.(91). In the following sections, it is assumed that trial functions \( \tilde{u} \) satisfy \( \tilde{u} = \bar{u} \) on \( S_u \).

### 2.1 Point collocation method

In the point collocation method, we select at least the same number of collocation points as the unknown parameters and determine the parameters \( \hat{u}^J \) such that the residual is zero at the selected points.

The collocation method is equivalent to the residual method in which the weight function is chosen as

\[
\hat{u}^J = f_I,
\]

or

\[
K_{IJ} \hat{u}^J = f_I.
\]  \hspace{1cm} (92)

where \( K_{IJ} \) and the vector \( f_I \) in eq.(92) can be expressed by

\[
K_{IJ} = -\Gamma N^J(x) \text{ or } B N^J(x) \quad \text{and} \quad f_I = \bar{f}(x) \text{ or } \bar{t}(x)
\]  \hspace{1cm} (96)

### 2.2 Least squares method

In the least squares method, the sum of the squares of the residuals defined by

\[
I = \int_D R_D^2 dV + \int_{S_t} R_S^2 dS
\]  \hspace{1cm} (97)

is minimized.

The necessary condition to minimize \( I \) is

\[
\frac{\partial I}{\partial \hat{u}^I} = 0, \quad I = 1, 2, \ldots
\]  \hspace{1cm} (98)

yielding the system of equations:

\[
\int_D R_D^2 dV + \int_{S_t} R_S^2 dS = 0 (99)
\]

or

\[
\left[ \int_D \Gamma N^I(x) \cdot \Gamma N^J(x) dV + \int_{S_t} B N^I(x) \cdot B N^J(x) dS \right] \hat{u}^J
\]

\[
- \left[ \int_D \Gamma N^I(x) \cdot \bar{f}(x) dV + \int_{S_t} B N^I(x) \cdot \bar{t}(x) dS \right] = 0
\]  \hspace{1cm} (100)

As shown in the above equation, the least squares method is considered as a weighted residual method with the weight functions \(-\Gamma N^I(x)\) in the domain \( D \) and \( B N^I(x) \) on the boundary \( S_t \).
2.3 Classical Galerkin method

If only the domain integral in eq.(91) is taken into account and the shape functions $N^I(x)\hat{w}^I$ are used as the weight functions $w_I$, where $\hat{w}^I$ are arbitrary coefficients, then we have

$$
\sum_{I=1}^{M} \hat{w}^I \left[ \sum_{J=1}^{M} \left\{ -\int_{D} N^I(x) \Gamma N^J(x) dV \right\} \hat{u}^J - \left\{ \int_{D} N^I(x) f(x) dV \right\} \right] = 0 \quad (101)
$$

2.4 Galerkin method

In the classical Galerkin method, the boundary condition is not considered. Therefore, the classical Galerkin method is not convenient for practical use. Here the Galerkin method taking account of the boundary condition is formulated.

Consider the weighted residual as follows:

$$
\int_{D} w(x)(-\Gamma \hat{u}(x) - f(x)) dV + \int_{S_t} w(x)(B \hat{u}(x) - t(x)) dS = 0. \quad (102)
$$

Here we assume that $w$ is a weighted function satisfying $w = 0$ on $S_u$ and $\hat{u}$ satisfies the boundary condition $\hat{u} = \bar{u}$ on $S_u$.

Since $\Gamma = \nabla \cdot \Sigma$, the first integral in eq.(102) is written as

$$
\int_{D} w(x) \Gamma \hat{u}(x) dV = \int_{D} w(x) \nabla \cdot \Sigma \hat{u}(x) dV
$$

$$
= \int_{S_t} w(x) n \cdot \Sigma \hat{u}(x) dS - \int_{D} \nabla w(x) \cdot \Sigma \hat{u}(x) dV
$$

$$
= \int_{S_t} w(x) B \hat{u}(x) dS - \int_{D} \nabla w(x) \cdot \Sigma \hat{u}(x) dV \quad (103)
$$

where the divergence theorem is used. Note that in the above equation, the integral on $S_u$ vanishes due to $w = 0$ on $S_u$. Then eq.(102) becomes

$$
\int_{D} \nabla w(x) \cdot \Sigma \hat{u}(x) dV - \int_{D} w(x) f(x) dV - \int_{S_t} w(x) t(x) dS = 0. \quad (104)
$$

This equation has the weak form which involves the Neumann boundary condition on $S_t$, which can be solved under the Dirichlet boundary condition $\hat{u} = \bar{u}$ and the constraint condition $w = 0$ on $S_u$.

Now it is assumed that the trial function $\hat{u}$ is expressed by

$$
\hat{u}(x) = \sum_{J=1}^{M} N^J(x) \hat{u}^J + N^0(x) \bar{u} \quad (105)
$$

where $N^J(x)$ and $N^0(x)$ are the shape functions equal to zero and one on the boundary $S_u$, respectively. Also the weight function $w$ is selected as

$$
w(x) = N^I(x) \hat{w}^I \quad (106)
$$

where $N^I(x)$ is the same shape function used in eq.(105). Then the conditions $u = \hat{u}$ and $w = 0$ on $S_u$ are satisfied.

Substitution of eqs.(105) and (106) into eq.(104) yields

$$
\sum_{I=1}^{M} \hat{w}^I \left[ \sum_{J=1}^{M} \int_{D} \nabla N^J(x) \cdot \Sigma N^J(x) dV \hat{u}^J - \int_{D} N^I(x) f(x) dV \right] \left[ \sum_{J=1}^{M} \int_{D} \nabla N^J(x) \cdot \Sigma N^J(x) dV \hat{u}^J \right] = 0. \quad (107)
$$
Example 2.1 Consider the solution $u$ satisfying the equation

$$\frac{d^2u}{dx^2} + f_0 = 0 \quad 0 \leq x \leq l$$

subjected to the boundary conditions

$$u = 0 \quad \text{at } x = 0, \quad du/dx = f_0 l \quad \text{at } x = l.$$  \hfill (108)

The exact solution for the above problem is $u(x) = f_0 x(2l - x)/2$ and the value at $x = l/2, u(l/2)$, is obtained as $u(l/2) = f_0 l/2(2l - x/4) = f_0 l^2/8$.

Now the boundary value problem mentioned above is solved by means of both the classical Galerkin method and the Galerkin method assuming that the solution is approximated by $	ilde{u}(x) = N^1(x)\hat{u}^1$, where $N^1(x) = x(2l - x)$.

Note that the approximated function is slightly different from the exact solution.

**Classical Galerkin method** For $N^1(x) = x(2l - x)$ and $\Gamma N^1(x) = d^2N^1(x)/dx^2 = -2$, eq.(101) becomes

$$\int_0^l x(2l - x)(-2)dx\hat{u}^1 + \int_0^l x(2l - x)f_0 dx = 0$$

From eq.(110), we obtained $\hat{u}^1 = f_0/2$. The solution $\tilde{u}(x)$ is obtained by $\tilde{u}(x) = f_0 x(2l - x)/2$. At $x = l/2, \tilde{u}(l/2) = 3/8f_0 l^2$, which is much different from the exact solution $u(l/2) = 7/8f_0 l^2$.

**Galerkin method** From eq.(107), the following equation based on the Galerkin method is formulated.

$$\int_0^l dN^1 dx N^1 dx \hat{u}^1 - \left( \int_0^l N^1(x)f_0 dx + N^1(1)f_0 l \right) = 0$$

Substitution of $N^1(x) = x(2l - x)$ into the above equation yields

$$\int_0^l 2(l - x)2(l - x)dx\hat{u}^1 - \left( \int_0^l (2xl - x^2)f_0 dx + l(2l - l)f_0 l \right) = 0,$$

from which $\hat{u}^1 = 5f_0/4$. Hence, $\tilde{u}(x) = 5f_0 x(2l - x)/4$. Also at $x = l/2, \tilde{u}(l/2) = 15/16f_0 l^2$. Thus we can conclude that the Galerkin method with the boundary condition gives a value closer to the exact solution than the classical Galerkin method, in which no boundary condition is considered.

Example 2.2 Solve the governing equation

$$\frac{d^2u}{dx^2} + 1 = 0 \quad 0 \leq x \leq 1$$

by using the Galerkin method with the trial function $u(x) = \sum_{J=1}^2 N^J(x)\hat{u}^J$, where $N^J(x) = x^J$. The boundary conditions are given by $u(0) = 0$ and $du/dx(1) = 20$.

The Galerkin method yields the following equation for the above mentioned problem:

$$\sum_{I=1}^M \hat{u}^I \left[ \sum_{J=1}^M \int_0^l \left\{ \frac{dN^I}{dx} \frac{dN^J}{dx} \right\} dx \hat{u}^J \right] - \left( \int_0^l N^1(x)f_I dx - \int_0^l \frac{dN^I}{dx} \frac{dN^0}{dx} dx \hat{u}^I + N^I(1)f_I \right) = 0$$

(114)

where $f_I = 1, \hat{u} = 0, f = 20, M = 2, N^1(x) = x$ and $N^2(x) = x^2$. The calculation of the matrix $K_{IJ}$ and the vector $f_I$ leads to the equations

$$\begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix} \left\{ \hat{u}^1 \hat{u}^2 \right\} = \left\{ \begin{array}{c} 20 \frac{1}{2} \\ 20 \frac{1}{2} \end{array} \right\}$$

(115)

which can be solved for $\hat{u}^J$ ($J = 1, 2$).
Problem 2.1 Consider the governing equation
\[
\frac{d^2 u}{dx^2} - u = 0 \quad 0 \leq x \leq 1 \tag{116}
\]
subjected to the boundary condition \(u(0) = 0\) and \(du/dx(1) = 20\). Solve the problem by using the Galerkin method with the trial function \(u(x) = \sum_{j=1}^{N} N_j(x) \bar{u}_j\), where \(N_j(x) = x^j\).

Problem 2.2 Derive the weak form based on the Galerkin method for the multi-dimensional Laplace problem, for which the governing equation is given by
\[
\nabla \cdot \nabla u = -\bar{f} \text{ in } D \tag{117}
\]
and the boundary conditions are as follows.
\[
u = \bar{u} \text{ on } S_u \tag{118}
\]
\[
t \equiv n \cdot \nabla u = \bar{t} \text{ on } S_t = S \setminus S_u \tag{119}
\]

Problem 2.3 Solve the same problem as Problem 1.9.1 using the weak form.