1.7 Natural boundary condition

So far, the stationary problem for $J[u] = \int_a^b F(x,u,u')dx$ has been solved, assuming that the boundary condition for $u$ is given at $x = a$ and $b$, i.e., $\delta u(a) = \delta u(b) = 0$. Now we consider the same stationary problem without specifying any boundary condition for $u$ in advance.

The first variation of $J[u]$ is written as

$$\delta J[u] = \int_a^b [F_u(x,u,u') - \frac{d}{dx}F_{u'}(x,u,u')]\delta u(x)dx + F_{u'}(x,u,u')\delta u(x)|_a^b = 0 \quad (49)$$

From this equation, we have

$$F_u(x,u,u') - \frac{d}{dx}F_{u'}(x,u,u') = 0 \quad \text{for} \quad a < x < b, \quad (50)$$

$$\delta u = 0 \quad \text{or} \quad F_{u'}(x,u,u') = 0, \quad \text{at} \quad x = a, \quad (51)$$

$$\delta u = 0 \quad \text{or} \quad F_{u'}(x,u,u') = 0, \quad \text{at} \quad x = b. \quad (52)$$

The second equations in eqs.(51) and (52) are called the natural boundary condition, while the first equations for $u(x)$ at $x = a$ and $b$ are called the rigid boundary condition.

If the natural boundary condition is given in the form of $F_{u'}(x,u,u')|_{x=a} = \alpha$ and $F_{u'}(x,u,u')|_{x=b} = \beta$, the functional $J[u]$ to be minimized is modified as follows.

$$J[u] = \int_a^b F(x,u,u')dx - \{\beta u(b) - \alpha u(a)\} \quad (53)$$

**Problem 1.7.1** Obtain the natural boundary conditions for the functional $J[u] = \int_a^b F(x,u,u',u'')dx$.

**Answer:** $\partial F/\partial u' - d(\partial F/\partial u'')/dx = 0$, $\partial F/\partial u'' = 0$ at $x = a, b$

1.8 Variational theorem for linear elasticity

Consider the domain $D$ enclosed by the boundary $S$, occupied by a linearly elastic solid. Let $u, \sigma, \epsilon$ be the displacement, stress, and strain, respectively. Also let $\vec{f}, \vec{t}, \vec{u}$ be the given body force, surface traction and displacement, respectively. For a linearly elastic solid in equilibrium state, we have the following basic equations:

**Equilibrium equation:** $\nabla \cdot \sigma + \vec{f} = 0$, for $x \in D \quad (54)$

**Constitutive equation:** $\sigma = E: \epsilon$, for $x \in D \quad (55)$

**Strain-displacement relation:** $\epsilon = L \cdot u$, for $x \in D \quad (56)$

**Traction boundary condition:** $\vec{t} \equiv n \cdot \sigma = \vec{t}$, for $x \in S_t \quad (57)$

**Displacement boundary condition:** $u = \bar{u}$, for $x \in S_u = S \backslash S_t \quad (58)$

where $E$ is the elastic constant, $L$ is the differential operator having the components $L_{ijk} = (\delta_{ik}\partial/\partial x_j + \delta_{jk}\partial/\partial x_i)/2$, and $n$ is the normal vector on the boundary $S$.

The problem is to obtain $u, \sigma, \epsilon$ in $D$ by solving eqs.(54)-(56) under the boundary conditions (57) and (58).

1.8.1 Hu-Washizu variational theorem

The Hu-Washizu theorem states that all basic equations for linear theory of elasticity, shown in eqs.(54)-(58), can be obtained by minimizing the functional with independent parameters $u, \sigma$ and $\epsilon$.

$$J[u, \sigma, \epsilon] = \int_D \left\{ \frac{1}{2} \epsilon^T : E : \epsilon + \sigma^T : (L \cdot u - \epsilon) - \vec{u}^T \cdot \vec{f} \right\} dV$$

$$- \int_{S_t} \vec{u}^T \cdot \vec{t} dS - \int_{S_u} (\sigma \cdot n)^T \cdot (\bar{u} - \bar{u}) dS \quad (59)$$
1.8.2 Hellinger-Reissner variational theorem

If we substitute \( \epsilon = E^{-1}\sigma \) into eq.(60), then we have the functional with the parameters \( u \) and \( \sigma \).

\[
J[u, \sigma] = \int_D \left\{ -\frac{1}{2} \sigma^T : E^{-1} : \sigma + \sigma^T : L \cdot u - u^T \cdot \bar{f} \right\} dV
- \int_{S_i} u^T \cdot \bar{t} dS - \int_{S_n} (\sigma \cdot n)^T \cdot (u - \bar{u}) dS
\]

The Hellinger-Reissner variational approach is to minimize the functional \( J[u, \sigma] \) with the known strain-stress relation of \( \epsilon = E^{-1}\sigma \).

1.8.3 Minimum potential energy theorem

If we further introduce \( \sigma = E : \epsilon = E : L \cdot u \) in \( D \) and \( u = \bar{u} \) on \( S_n \), then the functional \( J[u] \) is obtained as follows.

\[
J[u] = \int_D \left\{ \frac{1}{2} (L \cdot u)^T : E : (L \cdot u) - u^T \cdot \bar{f} \right\} dV - \int_{S_i} u^T \cdot \bar{t} dS
\]

**Problem 1.8.1** Derive the basic equations in each theorem by minimizing the functional defined by eqs.(60), (61) or (62).

Hint: use the following equation in the derivation

\[
\int_D \sigma^T : L \cdot \delta u dV = \int_D \sigma^T : \nabla \delta u dV \quad \text{(due to stress symmetry)}
\]

(63)

\[
= \int_D \{ \nabla \cdot (\sigma \cdot \delta u) - (\nabla \cdot \sigma) \cdot \delta u \} dV
\]

(64)

\[
= \int_S n \cdot \sigma \cdot \delta u dS - \int_D (\nabla \cdot \sigma) \cdot \delta u dV
\]

(65)

**Problem 1.8.2** Obtain the basic equations for the beam in \( 0 < x < l \) by making stationary the total potential energy, given by

\[
U = \int_0^l \left\{ \frac{1}{2} EI \left( \frac{d^2 u}{dx^2} \right)^2 - q(x)u(x) \right\} dx - \left\{ -M_0 \left( \frac{du}{dx} \right)_0 + M_l \left( \frac{du}{dx} \right)_l + Q_0 u_0 - Q_l u_l \right\}
\]

(66)

where \( u, EI, q, M \) and \( Q \) are the deflection, the bending rigidity, the distributed load per length, the bending moment and the shear force, respectively, and the subscripts \( 0 \) and \( l \) indicate the boundary values at \( x = 0 \) and \( l \), respectively.

1.9 Trial function method — Ritz’s method

We consider the problem governed by the following Poisson’s equation

\[
\nabla^2 u(x, y) = \bar{f}(x, y) \quad \text{in} \quad D
\]

(67)

where \( u \) is the function to be determined and \( \bar{f} \) is a given function. The boundary condition is given by \( u = 0 \) on the boundary \( S \).

To solve eq.(67) is equivalent to find the function \( u \) by minimizing the functional

\[
J[u] = \int \int_D (u_x^2 + u_y^2 + 2\bar{f}u) dx dy
\]

(68)

Now the variational problem is approximately solved by assuming that the function \( u \) is expressed by using the trial function as follows:

\[
u(x, y) \approx c_1 w_1(x, y) + c_2 w_2(x, y) + \ldots + c_n w_n(x, y)
\]

(69)

where \( c_1, \ldots, c_n \) are the coefficients and \( w_1, \ldots, w_n \) are linearly independent functions called trial bases. Note that each trial basis \( w_i \) should satisfy \( w_i = 0 \) on the boundary \( S \).
Substituting eq.(69) into eq.(68), the functional $J[u]$ can be expressed in the following form

$$J[u(c_i)] \approx \sum_{i=1}^{n} \sum_{k=1}^{n} c_i c_k J_2[w_i, w_k] + 2 \sum_{k=1}^{n} c_k J_1[w_k] + J_0$$ (70)

where

$$J_2[u, v] = \int \int_D (u_x v_x + u_y v_y) dxdy$$ (71)
$$J_1[u] = \int \int_D \bar{f} u dxdy, \quad J_0 = 0$$ (72)

The procedure for minimizing $J[u]$ leads to

$$\frac{\partial J}{\partial c_i} = 0 \quad (i = 1, \ldots, n)$$ (73)

which produces the system of equations to determine $c_k$:

$$\sum_{k=1}^{n} c_k J_2[w_i, w_k] + J_1[w_i] = 0 \quad (i = 1, \ldots, n)$$ (74)

The above-mentioned method is called the Ritz’s method. The keypoint of the Ritz’s method is how to select appropriate functions for trial basis $w_1, \ldots, w_n$.

**Example 1.9.1** Consider the following boundary value problem:

$$x u'' + u' + \frac{x^2 - 1}{x} u + x^2 = 0, \quad 1 \leq x \leq 2$$ (75)

subjected to the boundary condition $u(1) = u(2) = 0$. The exact solution to the above equation is given by $u(x) = 3.6072 J_1(x) + 0.75195 Y_1(x) - x$, where $J_1$ and $Y_1$ are Bessel function and Neumann function of the first order, respectively. For example, we have $u(1.5) = 0.2024$.

The variational principle equivalent to the above boundary value problem is to find the solution to minimize the functional

$$J[u] = \frac{1}{2} \int_1^2 \left( (u')^2 - \frac{x^2 - 1}{x} u^2 - 2x^2 u \right) dx$$ (76)

Now we approximate the function $u$ by $u \approx c_1 w_1 = c_1 (x - 1)(2 - x)$. It then follows that

$$J[u(c_1)] = \frac{1}{2} \int_1^2 \left\{ (c_1 w_1')^2 - \frac{x^2 - 1}{x} (c_1 w_1)^2 - 2x^2 c_1 w_1 \right\} dx$$ (77)

$\frac{\partial J[u(c_1)]}{\partial c_1} = 0$ yields the following equation

$$c_1 \int_1^2 \left\{ (w_1')^2 - \frac{x^2 - 1}{x} (w_1)^2 \right\} dx - \int_1^2 x^2 w_1 dx = 0.$$ (78)

Hence we have

$$c_1 = \frac{\int_1^2 x^2 w_1 dx}{\int_1^2 \left\{ (w_1')^2 - \frac{x^2 - 1}{x} (w_1)^2 \right\} dx} = 0.8110.$$ (79)

So the approximated solution is $u \approx 0.81110 \times w_1(x)$, which gives $u(1.5) \approx 0.2027$.

**Problem 1.9.1** Consider the Laplace field satisfying $\nabla^2 u + 1 = 0$ in the 2D domain $0 \leq x \leq 1$, $0 \leq y \leq 1$, where the boundary condition is given by $u = 0$ on $x = 0, 1$, and $y = 0, 1$. Assuming $u(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sin(i \pi x) \sin(j \pi y) \tilde{u}_{ij}$, determine the coefficients $\tilde{u}_{ij}$.

**Problem 1.9.2** Consider the beam with the unit length subjected to the distributed load per length of $q(x) = q_0(1-x)$ and the following boundary conditions

$$u(0) = 0 \quad \text{and} \quad u'(0) = 0 \quad \text{at} \quad x = 0$$ (80)
$$u(1) = 0 \quad \text{and} \quad M(1) = -EIu''(1) = 0$$ (81)
where \( u \), \( M \) and \( EI \) are the deflection, the bending moment and the bending rigidity of the beam, respectively. For such a beam, the total potential energy \( U \) is given by

\[
U = \int_0^1 \left( \frac{1}{2} \frac{M^2}{EI} - \bar{q}u \right) dx = \int_0^1 \left( \frac{EI}{2} (u'')^2 - \bar{q}u \right) dx. \tag{82}
\]

Assuming that the deflection \( u \) is approximated by \( u \approx c_1 w_1 + c_2 w_2 \), where

\[
w_1 = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \quad \text{and} \quad w_2 = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + b_5 x^4,
\]

determine the deflection \( u \).

**Hint** Using the rigidity boundary conditions of

\[
w_1(0) = w_2(0) = 0, \quad w_1'(0) = w_2'(0) = 0 \quad \text{and} \quad w_1(1) = w_2(1) = 0,
\]

show that \( w_1 \) and \( w_2 \) are obtained as

\[
w_1 = (x^2 - x^3) \quad \text{and} \quad w_2 = x^2 - 2x^3 + x^4.
\]

Then apply the variational principle by substituting

\[
u = c_1 w_1 + c_2 w_2
\]

into eq.(82).