3. THE CONCEPT OF STRESS

Here a deformable body during a finite motion is considered. Since the current configuration of many problems is not known, it is not convenient to work with stress tensors which are expressed in terms of spatial coordinates. For some cases it is more convenient to work with stress tensors that are referred to the reference configuration or an intermediate configuration.

3.1 Traction vectors and stress tensors

Here surface tractions which are related to either the current or the reference configuration are introduced. These surface tractions are expressed in terms of unique stress fields acting on a normal vector to a plane surface.

Surface tractions

Consider a deformable body \( B \) occupying an arbitrary region \( \Omega \) of physical space with boundary surface \( \partial \Omega \) at time \( t \).

It is postulated that arbitrary forces act on parts or the whole of the boundary surface (called external forces), and on an (imaginary) surface within the interior of that body (called internal forces) in some distributed manner.

Let the body now be cut by a plane surface which passes any given point \( x \in \Omega \) with spatial coordinates \( x \) at time \( t \). Since interaction of the two portions are considered, forces are transmitted across the internal plane surface. The infinitesimal resultant (actual) force acting on a surface element is denoted by \( dF \). Note the distributed so-called resultant couples not occurring in the classical formulation of continuum mechanics are omitted.
For every surface element
\[ df = tdS = TdS, \]  
(3.1)
\[
\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}), \quad \mathbf{T} = \mathbf{T}(X, t, \mathbf{N}). \quad \text{Cauchy's postulate}
\]
(3.2)
\[ t = t(x, t, n), \quad T = T(X, t, N). \]

Here,

- \( \mathbf{t} \), the Cauchy (or true) traction vector (force measured per unit surface area defined in the current configuration), acting on \( ds \) with outward normal \( \mathbf{n} \).
- \( \mathbf{T} \), the first Piola-Kirchhoff (or nominal) traction vector (force measured per unit surface area defined in the reference configuration), and points in the same direction as the Cauchy traction vector \( \mathbf{t} \). The (pseudo) traction vector \( \mathbf{T} \) does not describe the actual intensity. It acts on the region \( \Omega \) and is, in contrast to the Cauchy traction vector \( \mathbf{t} \), a function of the referential position \( \mathbf{X} \) and the outward normal \( \mathbf{N} \) to the boundary surface \( \partial \Omega_0 \). (Hence, the dashed line for \( \mathbf{T} \) in the figure.)

The vectors \( \mathbf{t} \) and \( \mathbf{T} \) that act across the surface elements \( ds \) and \( dS \) with respective normals \( \mathbf{n} \) and \( \mathbf{N} \) are referred to as surface tractions or as contact forces, stress vectors or just loads. Typical surface tractions are contact and friction forces or are caused by liquids or gases.

**Cauchy's stress theorem**

There exists unique second-order tensor fields \( \sigma \) and \( P \) so that,
\[
\begin{align*}
\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = \sigma(\mathbf{x}, t)\mathbf{n}, & \quad t_i = \sigma_{ij}n_j \\
\mathbf{T}(X, t, \mathbf{N}) = P(X, t)\mathbf{N}, & \quad T_i = P_{ik}N_k
\end{align*}
\]  
(3.3.1)

\( \sigma \) is a symmetric spatial tensor field called the Cauchy (or true) stress tensor.

\( P \) is a tensor field called the first Piola-Kirchhoff (or nominal) stress tensor, (sometimes simply called the Piola stress).

(Proof is omitted.)

\[ t(x, t, n) = -t(x, t, -n), \quad \mathbf{T}(X, t, \mathbf{N}) = -\mathbf{T}(X, t, -\mathbf{N}). \quad \text{Newton's (third) law of action and reaction} \]
(3.4)

The index notation in eqn (3.3.2) indicates that \( P \), is a two-point tensor in which one index describes spatial coordinates \( x_i \), and the other material coordinates \( X_r \). In section 4.3, pg 147 (GAH) it is established that \( \sigma \) is symmetric under the assumption that resultant couples are neglected.

Equation (3.3) which relates the surface traction with the stress tensor, is one of the most important axioms in continuum mechanics and is known as Cauchy's stress theorem (or as Cauchy's law). Basically, it states that if traction vectors such as \( \mathbf{t} \) or \( \mathbf{T} \) depend on the outward unit normals \( \mathbf{n} \) or \( \mathbf{N} \), then the traction vectors must be linear in \( \mathbf{n} \) or \( \mathbf{N} \), respectively.

An immediate consequence of eqn (3.3) are the following relationships,
Equation (3.3.1) in convenient matrix notation (useful for computational purposes) is,

\[
\{ t \} = [\sigma] \{ n \},
\]  

(3.5)

\[
\{ t \} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \quad [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \quad \{ n \} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},
\]  

(3.6)

where \([\sigma]\) is the Cauchy stress matrix.

Relation between the Cauchy stress tensor \(\sigma\) and first Piola-Kirchhoff stress tensor \(P\).

From eqns (3.1) and (3.2): \(t(x,t,n)ds = T(X,t,N)ds\).

Using eqn (3.3),

\[
\sigma(x,t)n ds = P(X,t)N ds.
\]  

(3.7)

Using Nanson's formula \(nds = J G N dS\) (eqn (2.2.18) RWO),

\[
\sigma J G N dS = P N dS.
\]

i.e.,

\[
P = J \sigma G, \quad P_{ir} = J \sigma_{ij} G_{jr}.
\]  

(3.8)

The passage from \(\sigma\) to \(P\) and back is known as the Piola transformation (cf. definition on pg. 83 GAH). Strictly speaking, in order to obtain the two-point tensor \(P\), with components \(P_{ir}\), we have performed a Piola transformation on the second index of tensor \(\sigma\), with components \(\sigma_{ij}\).

From eqn (3.8),

\[
\sigma = \frac{1}{J} P F^T = \sigma^T, \quad \sigma_{ij} = \frac{1}{J} P_{ir} F_{jr} = \sigma_{ji},
\]  

(3.9)

which implies,

\[
P F^T = F P^T.
\]  

(3.10)

Hence, \(P\) is in general not symmetric.

**Example 3.1**  A deformation of a body is described by,

\[
x_1 = -6X_2, \quad x_2 = \frac{1}{2}X_1, \quad x_3 = \frac{1}{3}X_3.
\]  

(3.11)

The Cauchy stress tensor for a certain point of a body is given by its matrix representation as,

\[
[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ kN/cm}^2.
\]  

(3.12)
Determine the Cauchy traction vector \( \mathbf{t} \) and the first Piola-Kirchhoff traction vector \( \mathbf{T} \) acting on a plane, which is characterized by the outward unit normal \( \mathbf{n} = \mathbf{e}_2 \) in the current configuration.

**Solution:** From eqn (3.11), calculate \( F_{ij} = \frac{\partial x_i}{\partial X_j} \),

\[
\mathbf{F} = \begin{bmatrix} 0 & -6 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad \mathbf{F}^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ \frac{1}{6} & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.
\]

(3.13)

and \( \det \mathbf{F} = J = 1 \). Components of the first Piola-Kirchhoff stress tensor from eqn (3.8)

\[
\mathbf{P} = J \mathbf{[\boldsymbol{\sigma}]} \mathbf{[G]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ kN/cm}^2.
\]

(3.14)

Determine \( \mathbf{N} \): Nanson's formula \( \mathbf{n} \mathbf{d}s = J \mathbf{G} \mathbf{N} \mathbf{d}s \Rightarrow \mathbf{N} \mathbf{d}s = \frac{1}{J} \mathbf{F}^T \mathbf{n} \mathbf{d}s \),

\[
\mathbf{N} \mathbf{d}s = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -6 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \mathbf{d}s = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \mathbf{d}s = \frac{1}{2} \mathbf{e}_2 \mathbf{d}s \Rightarrow \mathbf{N} = \mathbf{e}_1.
\]

(3.15)

Finally using Cauchy's stress theorem,

\[
\{ \mathbf{t} \} = \mathbf{\sigma} \{ \mathbf{n} \} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \\ 0 \end{bmatrix} \text{ kN/cm}^2,
\]

\[
\{ \mathbf{T} \} = \mathbf{P} \{ \mathbf{N} \} = \begin{bmatrix} 0 & 0 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \text{ kN/cm}^2.
\]

(3.15)

i.e., \( \mathbf{t} = 50 \mathbf{e}_2 \) and \( \mathbf{T} = 100 \mathbf{e}_2 \); \( \mathbf{t} \) and \( \mathbf{T} \) have the same direction (as expected by eqn (3.1)). The magnitude of \( \mathbf{T} \) is twice that of \( \mathbf{t} \), because from eqn (3.15), the deformed area is twice the undeformed area.