3. Strong and Weak Forms for 1-D Problems

3.0 Introduction

The strong form consists of the governing equations and boundary conditions for a given physical problem. The governing equations are usually partial differential equations but in the 1-D case the governing equations are ordinary differential equations. The weak form is an integral form of these equations, which is needed to formulate the finite element method (FEM).

The approximation functions are combined with the weak form to obtain the discrete finite element (FE) eqns.

3.1 The strong form for an axially loaded bar

Consider the static response of an elastic bar of length \( l \) and variable cross-section \( A(x) \), subjected to a body force or distributed loading \( b(x) \) per unit length, (i.e., the units of \( b(x) \) are force/length). In addition loads called tractions \( \bar{T} \) with units of force per area, can be prescribed at the ends of the bar, where the displacement is not prescribed.

The elastic bar must satisfy the following conditions:
1. Equilibrium.
2. Hooke’s law in 1-D, i.e., \( \sigma(x) = E(x)\varepsilon(x) \), where \( \sigma(x) \) - stress, \( E(x) \) - Young’s modulus, \( \varepsilon(x) \) - strain.
3. Displacement field must be compatible.
4. The strain-displacement relationship.
The differential eqn of the bar is obtained from equilibrium of internal force \( p(x) \) and external force \( b(x) \) acting on the body in the \( x \)-direction.

\[
+ \sum F_x = 0: \quad -p(x) + b(x + \Delta x/2)\Delta x + p(x + \Delta x) = 0,
\]

i.e.,
\[
\frac{p(x + \Delta x) - p(x)}{\Delta x} + b(x + \Delta x/2) = 0.
\]

Taking the limit of the above eqn as \( \Delta x \to 0 \),
\[
\frac{dp(x)}{dx} + b(x) = 0. \tag{3.1}
\]

The nominal stress of the bar is defined by,
\[
\sigma(x) = \frac{p(x)}{A(x)}. \tag{3.2}
\]

The strain-displacement relation (or kinematic eqn) in 1-D is,
\[
\varepsilon(x) = \lim_{\Delta x \to 0} \frac{\text{elongation}}{\text{original length}} = \frac{u(x + \Delta x) - u(x)}{\Delta x} = \frac{du(x)}{dx}. \tag{3.3}
\]

The stress-strain law for a linear elastic material is Hooke’s law, which in 1-D is,
\[
\sigma(x) = E(x)\varepsilon(x). \tag{3.4}
\]

Substituting eqns (3.2)-(3.4) into eqn (3.1),
\[
\frac{d}{dx} \left( E(x)A(x)\frac{du(x)}{dx} \right) + b(x) = 0, \quad 0 < x < l. \tag{3.5}
\]

The differential eqn (3.5) is a specific form of the equilibrium eqn (3.1). Equation (3.1) is valid for both linear and nonlinear materials, whereas eqn (3.5) assumes linearity in the definition of strain (eqn (3.3)) and the stress-strain law (eqn (3.4)).

Compatibility is satisfied by requiring the displacement to be continuous.
To solve the 2nd-order ODE (eqn (3.5)), two boundary conditions need to be prescribed. Consider the following boundary conditions,
at \( x = 0 \), the traction (i.e., force/area) \( \bar{T} \) is prescribed, and
at \( x = l \), the displacement \( u(l) \) is prescribed,

\[
\sigma(0) = \left( E \frac{du}{dx} \right)_{x=0} = \frac{p(0)}{A(0)} = -\bar{T}, \quad (3.6a)
\]
\[
u(l) = \bar{u}. \quad (3.6b)
\]

(The superposed bar indicates a prescribed boundary value.)

The traction \( \bar{T} \) is positive when it acts in the positive \( x \)-direction regardless of which face it is acting on, whereas the stress is positive in tension and negative in compression, so that on a negative face, a positive stress corresponds to a negative traction.

Note: Either the load or the displacement can be specified at a boundary but not both at the same time.

The governing differential eqn (3.5), together with the boundary conditions eqn (3.6), is the strong form of the problem,

\[
\frac{d}{dx} \left( E(x)A(x) \frac{du(x)}{dx} \right) + b(x) = 0, \quad 0 < x < l,
\]
\[
\sigma(0) = \left( E \frac{du}{dx} \right)_{x=0} = \frac{p(0)}{A(0)} = -\bar{T}, \quad (3.7)
\]
\[
u(l) = \bar{u}.
\]

(In eqn (3.7), \( b(x), \bar{T} \) and \( \bar{u} \) are prescribed, the unknown is \( u(x) \)).