9.4 Vector and scalar functions and fields. Derivatives

We now begin with vector calculus which concerns two kinds of functions:

**Vector functions**: Functions whose values are vectors depending on the points \( P \) in space,

\[
v = v(P) = v_1(P)\mathbf{i} + v_2(P)\mathbf{j} + v_3(P)\mathbf{k}.
\]

**Scalar functions**: Functions whose values are scalars depending on the points \( P \) in space,

\[
f = f(P).
\]

Here, \( P \) is a point in the domain of definition, which in applications is a 3-D domain or a surface or a curve in space. A vector function defines a vector field and a scalar function defines a scalar field in that domain or on that surface or curve. Examples of vector fields are field of tangent vectors of a curve, field of normal vectors of a surface, velocity field of a rotating body and the gravitational field (see Figs. 193-196). Examples of scalar fields are the temperature field in a body or the pressure field of the air in the earth's atmosphere.

Vector and scalar fields may also depend on time \( t \) or on some other parameters.

If we introduce Cartesian coordinates \( x, y, z \), then instead of \( v(P) \) we can write,

\[
v = v(x, y, z) = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}.
\]

Note that the components depend on the choice of the coordinate system, whereas a vector field that has a physical or a geometric meaning should have magnitude and direction depending only on \( P \), not on the choice of coordinate system. Similarly for the value of a scalar field \( f(P) = f(x, y, z) \).

Example 1: Scalar function (Euclidean distance in space)

The distance \( f(P) \) of any point \( P \) from a fixed point \( P_0 \) in space is a scalar function whose domain of definition is the whole space. \( f(P) \) defines a scalar field in space. If a Cartesian coordinate system is introduced and \( P_0 \) has the coordinates \( x_0, y_0, z_0 \) and \( P \) has the coordinates \( x, y, z \) then \( f \) is given by,

\[
f(P) = f(x, y, z) = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2}.
\]

If the given Cartesian coordinate system is replaced by another Cartesian coordinate system obtained by translating and rotating the given system, then the values of the coordinates of \( P \) and \( P_0 \) will in general change, but \( f(P) \) will have the same value as before. Hence \( f(P) \) is a scalar function.
Example 2: Vector field (Velocity field)
At any instant the velocity vectors $\mathbf{v}(P)$ of a rotating body $B$ constitutes a vector field, called the *velocity field* of the rotation. If a Cartesian coordinate system having the origin on the axis of rotation is introduced then (see Ex. 5, Sec. 9.3),

$$\mathbf{v}(x, y, z) = \mathbf{w} \times \mathbf{r} = \mathbf{w} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}),$$

(9.4.1)

where $x, y, z$ are the coordinates of any point $P$ of $B$ at the instant under consideration. If the coordinate system is such that the $z$-axis is the axis of rotation and $\mathbf{w}$ points in the positive $z$-direction, then $\mathbf{w} = \omega \mathbf{k}$ and

$$\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 0 & \omega \\
x & y & z
\end{vmatrix} = \omega(-yi + xj).$$

An example of a rotating body and the corresponding velocity field are shown in Fig. 195.

![Fig. 195. Velocity field of a rotating body](image)

Example 3: Vector field (Field of force, gravitational field)
A particle $A$ of mass $M$ is fixed to a point $P_0$ and a particle $B$ of mass $m$ is free to take up various positions $P$ in space. Then particle $A$ attracts particle $B$. According to *Newton's law of gravitation* the corresponding gravitational force $\mathbf{p}$ is directed from $P$ to $P_0$, and its magnitude is proportional to $1/r^2$, where $r$ is the distance between $P$ and $P_0$,

$$|\mathbf{p}| = \frac{c}{r^2},$$

(9.4.2)

where $c = GMm$ and $G = 6.67 \times 10^{-8} \text{ cm}^3/(\text{gm sec}^2)$ is the gravitational constant. Hence $\mathbf{p}$ defines a vector field in space. If a Cartesian coordinate system is introduced such that $P_0$ has the coordinates $x_0, y_0, z_0$ and $P$ has the coordinates $x, y, z$ then,

$$r = \left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{1/2}.$$

Assuming that $r > 0$ and introducing the vector,

$$\mathbf{r} = (x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k},$$
it is seen that $|\mathbf{r}| = r$ and $(-1/r)\mathbf{r}$ is a unit vector in the direction of $\mathbf{p}$; the minus sign indicated that $\mathbf{p}$ is directed from $P$ to $P_0$. From this and eqn (9.4.2),

$$
\mathbf{p} = |\mathbf{p}| \left(-\frac{1}{r}\right)\mathbf{r} = -\frac{c}{r^3} \mathbf{r} = -\frac{c}{r^3} [(x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}].
$$

(9.4.3)

This vector function describes the gravitational force acting on $B$.

![Figure 196. Gravitational field in Example 3](image)

### 9.4.1 Vector calculus

Next it is shown that the basic concepts of calculus, i.e., convergence, continuity, and differentiability can be defined for vector functions in a simple and natural way. Most important here is the derivative.

**Convergence:** An infinite sequence of vectors $\mathbf{a}_n$, $n = 1, 2, \ldots$, is said to converge if there is a vector $\mathbf{a}$ such that

$$
\lim_{n \to \infty} |\mathbf{a}_n - \mathbf{a}| = 0.
$$

(9.4.4)

Here $\mathbf{a}$ is called the limit vector of that sequence, and we write,

$$
\lim_{n \to \infty} \mathbf{a}_n = \mathbf{a}.
$$

(9.4.5)

Cartesian coordinates being given, this sequence of vectors converge to $\mathbf{a}$ if and only if the three sequences of components of the vectors converge to the corresponding components of $\mathbf{a}$.

Similarly, a vector function $\mathbf{v}(t)$ of a real variable $t$ is said to have the limit $I$ as $t$ approaches $t_0$, if $\mathbf{v}(t)$ is defined in some neighborhood of $t_0$ (possibly except at $t_0$) and

$$
\lim_{t \to t_0} |\mathbf{v}(t) - I| = 0.
$$

(9.4.6)

Then we write,

$$
\lim_{t \to t_0} \mathbf{v}(t) = I.
$$

(9.4.7)
Here a *neighborhood* of \( t_0 \) is an interval (segment) on the \( t \)-axis containing \( t_0 \) as an interior point (not as an endpoint).

**Continuity:** A vector function \( \mathbf{v}(t) \) is said to be *continuous* at \( t = t_0 \) if it is defined in some neighborhood of \( t_0 \) (including at \( t_0 \) itself) and

\[
\lim_{t \to t_0} \mathbf{v}(t) = \mathbf{v}(t_0). \quad (9.4.8)
\]

If a Cartesian coordinate system is introduced, we may write,

\[
\mathbf{v}(t) = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}.
\]

Then \( \mathbf{v}(t) \) is continuous at \( t_0 \) if and only if its three components are continuous at \( t_0 \).

**Derivative of a vector function:** A vector function \( \mathbf{v}(t) \) is said to be *differentiable* at a point \( t \) if the following limit exists,

\[
\mathbf{v}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}. \quad (9.4.9)
\]

This vector \( \mathbf{v}'(t) \) is called the *derivative* of \( \mathbf{v}(t) \). See fig. 197.

In components with respect to a given Cartesian coordinate system,

\[
\mathbf{v}'(t) = v_1'(t)\mathbf{i} + v_2'(t)\mathbf{j} + v_3'(t)\mathbf{k}. \quad (9.4.10)
\]

Hence the derivative \( \mathbf{v}'(t) \) is obtained by differentiating each component separately. For instance, if \( \mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j} \) then \( \mathbf{v}'(t) = \mathbf{i} + 2t\mathbf{j} \).

Equation (9.4.10) follows from (9.4.9) and conversely because (9.4.9) is a "vector form" of the usual formula for calculus by which the derivative of a function of a single variable is defined. (The curve in Fig. 197 is the locus of the terminal points representing \( \mathbf{v}(t) \) for values of the independent variable in some interval containing \( t \) and \( t + \Delta t \) in (9.4.9)).

The familiar differentiation rules continue to hold for differentiating vector functions, for instance,

\[
(c \mathbf{v})' = c \mathbf{v}',
\]

\[
(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}',
\]

and in particular,
Example 4: Derivative of a vector function of constant length

Let \( \mathbf{v}(t) \) be a vector function whose length is constant, say, \( |\mathbf{v}(t)| = c \). Then

\[
|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = c^2, \quad \text{and} \quad (\mathbf{v} \cdot \mathbf{v})' = 2\mathbf{v} \cdot \mathbf{v}' = 0,
\]
by differentiation. Hence, the derivative of a vector function \( \mathbf{v}(t) \) of constant length is either the zero vector or is perpendicular to \( \mathbf{v}(t) \).

\[ \text{Example 4: Derivative of a vector function of constant length} \]

9.4.2 Partial derivatives of a vector function

Partial differentiation of vector functions of two or more variables can be introduced as follows. Suppose that the components of a vector function,

\[
\mathbf{v}(t_1, \ldots, t_n) = v_1(t_1, \ldots, t_n)\mathbf{i} + v_2(t_1, \ldots, t_n)\mathbf{j} + v_3(t_1, \ldots, t_n)\mathbf{k},
\]

are differentiable functions of \( n \) variables \( t_1, \ldots, t_n \). Then the partial derivative of \( \mathbf{v} \) with respect to \( t_m \) is denoted by \( \partial \mathbf{v} / \partial t_m \) and is defined as the vector function,

\[
\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m} \mathbf{i} + \frac{\partial v_2}{\partial t_m} \mathbf{j} + \frac{\partial v_3}{\partial t_m} \mathbf{k}.
\]

Similarly, second partial derivatives are

\[
\frac{\partial^2 \mathbf{v}}{\partial t_m \partial t_n} = \frac{\partial^2 v_1}{\partial t_m \partial t_n} \mathbf{i} + \frac{\partial^2 v_2}{\partial t_m \partial t_n} \mathbf{j} + \frac{\partial^2 v_3}{\partial t_m \partial t_n} \mathbf{k},
\]

and so on.

Example 5: Partial derivatives

Let \( \mathbf{r}(t_1, t_2) = a \cos t_1 \mathbf{i} + a \sin t_1 \mathbf{j} + t_2 \mathbf{k} \). Then \( \frac{\partial \mathbf{r}}{\partial t_1} = -a \sin t_1 \mathbf{i} + a \cos t_1 \mathbf{j} \) and \( \frac{\partial \mathbf{r}}{\partial t_2} = \mathbf{k} \).

Various physical and geometric applications of derivatives of vector functions will be discussed in the next sections and in Chapter 10.