9.4 Vector and scalar functions and fields. Derivatives

We now begin with vector calculus which concerns two kinds of functions:

Vector functions: Functions whose values are vectors depending on the points P in space,

$$\mathbf{v} = \mathbf{v}(P) = v_1(P)\mathbf{i} + v_2(P)\mathbf{j} + v_3(P)\mathbf{k}\ .$$

Scalar functions: Functions whose values are scalars depending on the points P in space,

$$f = f(P)\,.$$

Here, P is a point in the domain of definition, which in applications is a 3-D domain or a surface or a curve in space. A vector function defines a *vector field* and a scalar function defines a *scalar field* in that domain or on that surface or curve. Examples of vector fields are field of tangent vectors of a curve, field of normal vectors of a surface, velocity field of a rotating body and the gravitational field (see Figs. 193-196). Examples of scalar fields are the temperature field in a body or the pressure field of the air in the earth's atmosphere.

Vector and scalar fields may also depend on time t or on some other parameters.

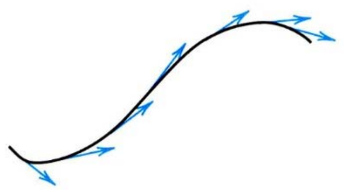

Fig. 193. Field of tangent vectors of a curve

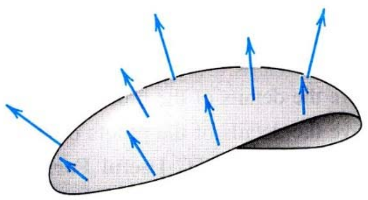

Fig. 194. Field of normal vectors of a surface

If we introduce Cartesian coordinates x, y, z, then instead of $\mathbf{v}(P)$ we can write,

$$\mathbf{v} = \mathbf{v}(x, y, z) = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}\ .$$

Note that the components depend on the choice of the coordinate system, whereas a vector field that has a physical or a geometric meaning should have magnitude and direction depending only on P, not on the choice of coordinate system. Similarly for the value of a scalar field $f(P) = f(x, y, z)$.

Example 1: Scalar function (Euclidean distance in space)

The distance $f(P)$ of any point P from a fixed point P_0 in space is a scalar function whose domain of definition is the whole space. $f(P)$ defines a scalar field in space. If a Cartesian coordinate system is introduced and P_0 has the coordinates x_0, y_0, z_0 and P has the coordinates x, y, z then f is given by,

$$f(P) = f(x, y, z) = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}\ .$$

If the given Cartesian coordinate system is replaced by another Cartesian coordinate system obtained by translating and rotating the given system, then the values of the coordinates of P and P_0 will in general change, but $f(P)$ will have the same value as before. Hence $f(P)$ is a scalar function.

Example 2: Vector field (Velocity field)

At any instant the velocity vectors $\mathbf{v}(P)$ of a rotating body B constitutes a vector field, called the *velocity field* of the rotation. If a Cartesian coordinate system having the origin on the axis of rotation is introduced then (see Ex. 5, Sec. 9.3),

$$\mathbf{v}(x,y,z) = \mathbf{w}\times\mathbf{r} = \mathbf{w}\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}), \tag{9.4.1}$$

where x, y, z are the coordinates of any point P of B at the instant under consideration. If the coordinate system is such that the z-axis is the axis of rotation and $\mathbf{w}$ points in the positive z-direction, then $\mathbf{w} = \omega\mathbf{k}$ and

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega(-y\mathbf{i} + x\mathbf{j}).$$

An example of a rotating body and the corresponding velocity field are shown in Fig. 195.

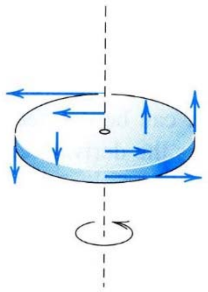

Fig. 195. Velocity field of a rotating body

Example 3: Vector field (Field of force, gravitational field)

A particle A of mass M is fixed to a point P_0 and a particle B of mass m is free to take up various positions P in space. Then particle A attracts particle B. According to *Newton's law of gravitation* the corresponding gravitational force $\mathbf{p}$ is directed from P to P_0, and its magnitude is proportional to $1/r^2$, where r is the distance between P and P_0,

$$|\mathbf{p}| = \frac{c}{r^2}, \tag{9.4.2}$$

where $c = GMm$ and $G = 6.67\times10^{-8}\,\mathrm{cm}^3/(\mathrm{gm\,sec}^2)$ is the gravitational constant. Hence $\mathbf{p}$ defines a vector field in space. If a Cartesian coordinate system is introduced such that P_0 has the coordinates x_0, y_0, z_0 and P has the coordinates x, y, z then,

$$r = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2}.$$

Assuming that $r > 0$ and introducing the vector,

$$\mathbf{r} = (x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k},$$

it is seen that $|\mathbf{r}| = r$ and $(-1/r)\mathbf{r}$ is a unit vector in the direction of $\mathbf{p}$; the minus sign indicated that $\mathbf{p}$ is directed from P to P_0. From this and eqn (9.4.2),

$$\mathbf{p} = |\mathbf{p}|(-\frac{1}{r})\mathbf{r} = -\frac{c}{r^3}\mathbf{r} = -\frac{c}{r^3}[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}]. \quad (9.4.3)$$

This vector function describes the gravitational force acting on B.

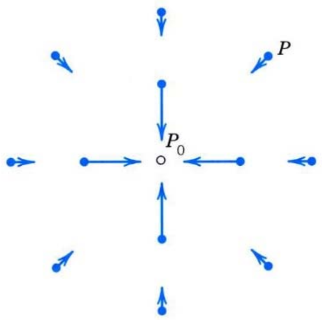

Fig. 196. Gravitational field in Example 3

9.4.1 Vector calculus

Next it is shown that the basic concepts of calculus, i.e., convergence, continuity, and differentiability can be defined for vector functions in a simple and natural way. Most important here is the derivative.

Convergence: An infinite sequence of vectors $\mathbf{a}_{(n)}$, $n = 1, 2, \ldots,$ is said to *converge* if there is a vector $\mathbf{a}$ such that

$$\lim_{n\to\infty} |\mathbf{a}_{(n)} - \mathbf{a}| = 0. \quad (9.4.4)$$

Here $\mathbf{a}$ is called the *limit vector* of that sequence, and we write,

$$\lim_{n\to\infty} \mathbf{a}_{(n)} = \mathbf{a}. \quad (9.4.5)$$

Cartesian coordinates being given, this sequence of vectors converge to $\mathbf{a}$ if and only if the three sequences of components of the vectors converge to the corresponding components of $\mathbf{a}$.

Similarly, a vector function $\mathbf{v}(t)$ of a real variable t is said to have the *limit* $\boldsymbol{l}$ as t approaches t_0, if $\mathbf{v}(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t\to t_0} |\mathbf{v}(t) - \boldsymbol{l}| = 0. \quad (9.4.6)$$

Then we write,

$$\lim_{t\to t_0} \mathbf{v}(t) = \boldsymbol{l}. \quad (9.4.7)$$

Here a *neighborhood* of t_0 is an interval (segment) on the t-axis containing t_0 as an interior point (not as an endpoint).

Continuity: A vector function $\mathbf{v}(t)$ is said to be *continuous* at $t = t_0$ if it is defined in some neighborhood of t_0 (including at t_0 itself) and

$$\lim_{t \to t_0} \mathbf{v}(t) = \mathbf{v}(t_0) . \tag{9.4.8}$$

If a Cartesian coordinate system is introduced, we may write,

$$\mathbf{v}(t) = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k} .$$

Then $\mathbf{v}(t)$ is continuous at t_0 if and only if its three components are continuous at t_0.

Derivative of a vector function: A vector function $\mathbf{v}(t)$ is said to be *differentiable* at a point t if the following limit exists,

$$\boxed{\mathbf{v}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} .} \tag{9.4.9}$$

This vector $\mathbf{v}'(t)$ is called the *derivative* of $\mathbf{v}(t)$. See fig. 197.

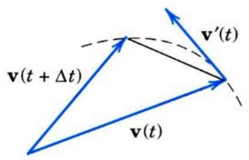

Fig. 197. Derivative of a vector function

In components with respect to a given Cartesian coordinate system,

$$\mathbf{v}'(t) = v_1'(t)\mathbf{i} + v_2'(t)\mathbf{j} + v_3'(t)\mathbf{k} . \tag{9.4.10}$$

Hence the derivative $\mathbf{v}'(t)$ is obtained by differentiating each component separately. For instance, if $\mathbf{v}(t) = t\,\mathbf{i} + t^2\mathbf{j}$ then $\mathbf{v}'(t) = \mathbf{i} + 2t\,\mathbf{j}$.

Equation (9.4.10) follows from (9.4.9) and conversely because (9.4.9) is a "vector form" of the usual formula for calculus by which the derivative of a function of a single variable is defined. (The curve in Fig. 197 is the locus of the terminal points representing $\mathbf{v}(t)$ for values of the independent variable in some interval containing t and $t + \Delta t$ in (9.4.9)).

The familiar differentiation rules continue to hold for differentaiating vector functions, for instance,

$$(c\,\mathbf{v})' = c\,\mathbf{v}',$$
$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}',$$

and in particular,

$$(\mathbf{u} \bullet \mathbf{v})' = \mathbf{u}' \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{v}' , \tag{9.4.11}$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' , \tag{9.4.12}$$

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})' = \mathbf{u}' \bullet (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \bullet (\mathbf{v}' \times \mathbf{w}) + \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}') . \tag{9.4.13}$$

Example 4: Derivative of a vector function of constant length

Let $\mathbf{v}(t)$ be a vector function whose length is constant, say, $|\mathbf{v}(t)| = c$. Then $|\mathbf{v}|^2 = \mathbf{v} \bullet \mathbf{v} = c^2$, and $(\mathbf{v} \bullet \mathbf{v})' = 2\mathbf{v} \bullet \mathbf{v}' = 0$, by differentiation. Hence, *the derivative of a vector function* $\mathbf{v}(t)$ *of constant length is either the zero vector or is perpendicular to* $\mathbf{v}(t)$.

9.4.2 Partial derivatives of a vector function

Partial differentiation of vector functions of two or more variables can be introduced as follows. Suppose that the components of a vector function,

$$\mathbf{v}(t_1,...,t_n) = v_1(t_1,...,t_n)\mathbf{i} + v_2(t_1,...,t_n)\mathbf{j} + v_3(t_1,...,t_n)\mathbf{k} ,$$

are differentiable functions of n variables $t_1,...,t_n$. Then the *partial derivative* of $\mathbf{v}$ with respect to t_m is denoted by $\partial\mathbf{v}/\partial t_m$ and is defined as the vector function,

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k} .$$

Similarly, second partial derivatives are

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k} ,$$

and so on.

Example 5: Partial derivatives

Let $\mathbf{r}(t_1,t_2) = a\cos t_1\,\mathbf{i} + a\sin t_1\,\mathbf{j} + t_2\,\mathbf{k}$. Then $\dfrac{\partial \mathbf{r}}{\partial t_1} = -a\sin t_1\,\mathbf{i} + a\cos t_1\,\mathbf{j}$ and $\dfrac{\partial \mathbf{r}}{\partial t_2} = \mathbf{k}$.

Various physical and geometric applications of derivatives of vector functions will be discussed in the next sections and in Chapter 10.